

Maxime Champion · Angel Alastuey · Thierry Dauxois · Stefano Ruffo

# Gravitation in the Microcanonical Ensemble: Appropriate Scaling Leading to Extensivity and Thermalization

Version: October 23, 2012

**Abstract** We introduce a simple model of hard spheres with gravitational interactions, for which we study a suitable scaling limit. Usual extensive properties are maintained notwithstanding the long range of gravitational interaction. We show that a local thermalization spontaneously emerges within a microcanonical description of the stationary state. In the considered scaling limit, the resulting density profile can be determined in a hydrostatic approach.

**Keywords** Gravitation · Microcanonical ensemble · Hard spheres · Extensivity

**PACS** 05.20.-y Classical statistical mechanics · 05.20.Gg Classical ensemble theory

## 1 Introduction

It is widely accepted that, in order to even attempt a statistical mechanics treatment of a system of  $N$  masses that mutually interact via Newtonian forces in three dimensions, one has to confine them inside a box of volume  $\Lambda$  to avoid evaporation. Confinement in a volume is a natural requirement also for a gas of particles interacting via short-range forces, but looks inappropriate for a model that should describe the Universe, a system that occupies full space. However, this seems a necessity for any thermodynamical theory of a self-gravitating system in the absence of cosmological expansion [1].

Moreover, a regularization of the interaction at short distances must be introduced to prevent the divergence of thermodynamic potentials [2,3,4,5,6]. In the case in which the Newtonian potential is softened at short distances and masses have no spatial extension, the minimal energy scales like  $N^2$  and H-stability is violated [7]. A mean-field (continuum) limit can, however, be performed by scaling the coupling constant as  $1/N$ , which yields an extensive energy [8]. It should be remarked that energy remains non additive and, as a consequence, one finds that different statistical ensembles give inequivalent predictions [9,10,5], leading to interesting phenomena like negative specific heat in the microcanonical ensemble [11,2]. If particles have instead a finite radius and cannot therefore penetrate each other, the minimal energy scales like  $N^{5/3}$  in three dimensions, violating extensivity.

In this paper we consider a self-gravitating system confined in a box and made of equal mass, equal shape and equal volume particles. Particles cannot penetrate each other as a result of a hard core short-range repulsion among them. As a consequence, collisions among the particles are elastic and, therefore, no aggregation phenomenon takes place. Particles collide elastically also with the walls of

---

Angel Alastuey · Maxime Champion · Thierry Dauxois  
 Laboratoire de Physique de l'Ecole Normale Supérieure de Lyon, Université de Lyon and CNRS, 46, allée d'Italie, F-69007 Lyon, France,

Stefano Ruffo  
 Dipartimento di Energetica “Sergio Stecco” and CSDC, Università di Firenze, CNISM and INFN, via S. Marta 3, 50139 Firenze, Italy

the confining box. Being then the dynamics energy and particle number conserving, it is appropriate to consider the system in the microcanonical ensemble. The microcanonical distribution in phase space is a stationary solution of Liouville equation, so that it indeed describes a stationary state of the system<sup>1</sup>. The hard core interaction prevents the divergence of the entropy. However, because of the long-range nature of the gravitational interaction, energy is *a priori* non extensive and should again scale like  $N^{5/3}$ . We will introduce a specific scaling limit in which we recover the extensivity of the energy. In that scaling limit, it turns out that fluctuations of the potential energy are small compared to the average when  $N \rightarrow \infty$ . Moreover, we show that if the energy per particle is large enough, local thermalization occurs due to hard sphere collisions. The one-body mass distribution is then found to obey a Boltzmann-like formula in terms of the mean-field gravitational potential, where the average kinetic energy plays the role of temperature. The fact that a local equilibrium is established, combined with a suitable separation of the length scales associated with hard core and gravitational interactions respectively, allows us to justify the introduction of a hydrostatic equation which express the balance between hard-sphere pressure and gravitational attraction.

We stress that the present scaling provides a finite mass density, while the ratio of the kinetic energy to the gravitational energy can take arbitrary values. Therefore, it should be well suited for describing a wide class of physical situations. Notice that other types of scaling have been considered in the literature [19,1], but they describe more specific situations like that of an infinitely diluted system for instance. Furthermore, once the scaling limit has been taken, the hard-core cutoff can be removed providing a meaningful limit as it would be desirable. In fact at large enough energy per particle we recover the well-known model of the isothermal gravitational sphere [11,17,18]. The situation is quite different from that studied in Ref. [3] for self-gravitating hard spheres in contact with a heat-bath in the canonical ensemble : when the size of the hard spheres is sent to zero while the other parameters are kept fixed, the equilibrium state becomes a Gaussian in velocity and a Dirac  $\delta$ -peak in position, in both two and three dimensions. Here, in our scaling limit, collapse is avoided because the gravitational interaction energy between two spheres at contact vanishes.

The paper is organized as follows. Section 2 presents the model and discuss the appropriate scaling continuous limit. The scaling properties of the potential energy are presented in Sec. 3, followed by the discussion of the mass distribution and the fluctuation of the potential energy. Section 4 presents how a Boltzmann-like formula for the inhomogeneous density emerges spontaneously, emphasizing a local thermalization in the microcanonical ensemble. The hydrostatic approach is then discussed in Sec. 5. We finally present our conclusions and draw some perspectives in Sec. 6.

## 2 Definitions

### 2.1 Hard spheres with gravitational interactions in the microcanonical ensemble

We consider a classical gravitational model made of  $N$  identical hard spheres with mass  $m$  and diameter  $\sigma$ , enclosed in a spherical box of volume  $\Lambda = 4\pi R^3/3$ . The corresponding Hamiltonian reads

$$H_N = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} v(|\mathbf{r}_i - \mathbf{r}_j|) \quad (1)$$

with the two-body interaction potential

$$v(r) = \infty \quad \text{for } r < \sigma, \quad v(r) = -Gm^2/r \quad \text{for } r > \sigma. \quad (2)$$

We consider that the previous system is isolated and does not exchange energy with some thermostat. Thus, its energy is fixed and equal to some value  $E$ . We assume that the corresponding stationary

---

<sup>1</sup> Use of the microcanonical distribution amounts to assume, roughly speaking, the hypothesis of molecular chaos leading to some ergodicity property. In fact, as argued in Refs. [13,14], gravitational systems may be trapped in metastable states through their dynamical evolution, while the corresponding transition times grow exponentially fast with respect to the number of particles. That mechanism might induce a breakdown of ergodicity, as suggested by results obtained from numerical simulations [15,16]. Therefore, as far as applications to real astrophysical systems are concerned, the microcanonical distribution has to be handled with some care.

state is described within the microcanonical ensemble. The corresponding distribution of positions and momenta of the  $N$  particles in the canonical phase space reads

$$f_{\text{micro}}(\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{p}_1, \dots, \mathbf{p}_N) = A_N \delta(E - H_N) , \quad (3)$$

while the total number of microstates is

$$\Omega(E, N, A) = A_N \int_{A^N \times R^{3N}} \prod_i d^3\mathbf{r}_i d^3\mathbf{p}_i \delta(E - H_N) , \quad (4)$$

where  $A_N$  is some normalization constant, which is not relevant for our purpose.

The microcanonical distribution  $f_{\text{micro}}$  remains stationary under time evolution generated by Hamiltonian  $H_N$ . We assume that there is no other conserved quantity than energy, like the total orbital momentum for instance [20]. We stress that  $\Omega(E, N, A)$  is finite for  $\sigma > 0$ , while it diverges for point particles with  $\sigma = 0$  for  $N \geq 3$  as shown in Ref. [6].

## 2.2 The scaling continuous limit

Since we are interested in the properties of a system with a large number of particles, it is useful to consider some limit where  $N \rightarrow \infty$ . For ordinary systems with short-range interactions, that limit is nothing but the usual thermodynamical limit where both the energy per particle  $E/N$  and the particle density  $\rho_p = N/A$  are kept fixed. In that limit, the physical parameters which describe one particle are also kept fixed. Because of both the attractive and long-range natures of gravitational interaction, such a limit would provide a collapsed state with non-extensive properties. In order to describe other physical situations of interest, other limits have been introduced in the literature, like the one describing an infinitely diluted system [19].

Here, we want to build a scaling limit when  $N \rightarrow \infty$ , which describes an infinite continuous fluid with the usual extensive properties. For that purpose, we consider that the parameters which define the particles vary with  $N$  like power laws, while the gravitational constant  $G$  is not rescaled contrarily to some mean-field scalings introduced in the literature [8]. Therefore, we set  $\sigma = d_0 N^\alpha$  and  $m = m_0 N^\delta$ , while the size  $R$  of the spherical box is chosen to diverge as  $\ell_0 N^\gamma$  with  $\gamma > 0$  so that the system indeed becomes infinitely extended. The particle density  $\rho_p = N/A$  behaves then as  $N^{1-3\gamma}$ . Since the particles are hard spheres, it is essential to keep the packing fraction  $\eta = \pi \rho_p \sigma^3 / 6$  bounded. By imposing that  $\eta$  remains constant, we find a first constraint

$$1 - 3\gamma + 3\alpha = 0 . \quad (5)$$

Moreover, we impose that the mass density  $\rho = m \rho_p$  remains also constant, so a second constraint arises,

$$1 - 3\gamma + \delta = 0 . \quad (6)$$

Eventually, the extensivity of the gravitational energy  $GM^2/R$  with the total mass  $M = Nm$ , provides the third constraint,

$$2 + 2\delta - \gamma = 1 . \quad (7)$$

The three exponents are then readily determined as  $\alpha = -2/15$ ,  $\delta = -2/5$  and  $\gamma = 1/5$ . Thus, the power laws defining the required scaling limit when  $N \rightarrow \infty$ , denoted SL, are

$$R = N^{1/5} \ell_0 , \quad m = N^{-2/5} m_0 , \quad \sigma = N^{-2/15} d_0 , \quad E = N \frac{G m_0^2}{\ell_0} \varepsilon \quad (8)$$

with parameters  $\ell_0$ ,  $m_0$ ,  $d_0$  and  $\varepsilon$  fixed.

Within the above SL, particles become infinitely small and light, while their inner mass density remains constant. Their number density  $\rho_p = (3/(4\pi\ell_0^3))N^{2/5}$  diverges, while the mass density  $\rho = m \rho_p = 3m_0/(4\pi\ell_0^3)$  is indeed kept fixed, as well as the packing fraction  $\eta = \pi \rho_p \sigma^3 / 6 = d_0^3/(8\ell_0^3)$ . That limit clearly describes an infinite continuous medium, with an extensive total energy  $E$ . The corresponding stationary state is controlled by two independent dimensionless parameters, namely  $\varepsilon$  which is the energy per particle in units of  $Gm_0^2/\ell_0$ , and the packing fraction  $\eta = d_0^3/(8\ell_0^3)$ . Notice that, within the considered scaling, the mean free path  $\ell = 1/(\pi\sqrt{2}\sigma^2\rho_p)$  remains proportional to the diameter  $\sigma$  of the hard spheres,  $\ell = \sigma/(6\sqrt{2}\eta)$ . Therefore, for high dilutions such that  $\eta \ll 1$ ,  $\ell$  is much larger than  $\sigma$ .

### 3 Scaling properties

#### 3.1 H-stability and extensivity of potential energy

For any allowed configuration, the potential energy

$$V_N = -\frac{1}{2} \sum_{i \neq j} \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (9)$$

is larger than that of the collapsed configuration where the  $N$  hard spheres make a single cluster with size  $L_{\text{coll}} \sim N^{1/3}\sigma$ , which is of order  $-Gm^2N^2/L_{\text{coll}}$ . In the scaling limit, the collapse radius  $L_{\text{coll}}$  diverges as  $N^{1/5}d_0$  and this provides the classical version of H-stability

$$V_N \geq -C_{\text{HS}} \frac{Gm_0^2}{d_0} N, \quad (10)$$

where  $C_{\text{HS}}$  is a positive real number entirely determined by the geometrical arrangement of the hard spheres at maximum packing. On the another hand, the potential energy should reach its maximum when all particles are homogeneously distributed on the spherical surface of the box. That maximum behaves as the gravitational energy of an empty sphere carrying the surface mass density  $Nm/(4\pi R^2)$ , so we expect the upper bound

$$V_N \leq -\frac{Gm_0^2}{2\ell_0} N. \quad (11)$$

According to bounds (10) and (11),  $V_N$  remains of order  $N$  for any configuration. This implies extensive lower and upper bounds for the average potential energy  $\langle V_N \rangle$ , where  $\langle A \rangle$  denotes the micro-canonical average of any observable  $A$  weighted by the distribution (3). Therefore, the nonextensive behaviours of the average potential energy encountered in other limits, like the usual thermodynamical limit for instance, do not occur in the present scaling limit. Accordingly, it seems quite plausible that the extensivity of  $\langle V_N \rangle$  is ensured here, namely

$$\lim_{\text{SL}} \frac{\langle V_N \rangle}{N} = \frac{Gm_0^2}{\ell_0} u(\varepsilon, \eta), \quad (12)$$

where  $u(\varepsilon, \eta)$ , the average potential energy per particle in units of  $Gm_0^2/\ell_0$ , is a well-behaved function of the intensive dimensionless parameters  $\varepsilon$  and  $\eta$ . We stress that a rigorous mathematical derivation of the extensive behaviour (12) is beyond our scope.

#### 3.2 Mass distributions

For further purposes, it is useful to introduce the  $n$ -body mass distributions. Let consider first the one-body mass distribution which reads,

$$\rho(\mathbf{r}) = m \left\langle \sum_i \delta(\mathbf{r}_i - \mathbf{r}) \right\rangle. \quad (13)$$

According to the scaling properties of the potential energy described above, we can reasonably expect that  $\rho(\mathbf{r})$  takes a well-defined shape in the SL. That statement can be made more precise as follows. For dimensional and spherical symmetry reasons, we can recast  $\rho(\mathbf{r})$  as

$$\rho(\mathbf{r}) = \rho(r) = \rho g_N^{(1)}(q; \varepsilon, \eta) \quad (14)$$

with  $q = r/R$  and where the function  $g_N^{(1)}(q; \varepsilon, \eta)$  is dimensionless. Notice that an explicit expression for  $g_N^{(1)}(q; \varepsilon, \eta)$  can be readily obtained in terms of multiple integrals upon the dimensionless positions  $\mathbf{q}_i = \mathbf{r}_i/R$ . Now, we make the assumption that  $g_N^{(1)}(q; \varepsilon, \eta)$  goes to some well-defined function  $g^{(1)}(q; \varepsilon, \eta)$  when  $N \rightarrow \infty$  with parameters  $q, \varepsilon, \eta$  fixed, so

$$\lim_{\text{SL}} \rho(qR) = \rho g^{(1)}(q; \varepsilon, \eta). \quad (15)$$

The physical content of that scaling property, as well as its limitations, will be discussed further.

The two-body mass distribution is

$$\rho^{(2)}(\mathbf{r}, \mathbf{r}') = m^2 \left\langle \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}) \delta(\mathbf{r}_j - \mathbf{r}') \right\rangle, \quad (16)$$

while similar definitions hold for higher order  $n$ -body mass distributions with  $n \geq 3$ . The scaling property (15) can be extended to such mass-distributions, namely

$$\lim_{\text{SL}} \rho^{(2)}(R\mathbf{q}, R\mathbf{q}') = \rho^2 g^{(2)}(\mathbf{q}, \mathbf{q}'; \varepsilon, \eta), \quad (17)$$

and so on. Of course, the limitations evocated above also apply to the present scaling behaviours.

Notice that the extensive behaviour (12) is consistent with the above scaling assumptions for the mass distributions. This is easily seen by starting from the integral expression of the average potential energy

$$\langle V_N \rangle = -\frac{1}{2} \int_{\Lambda^2} d^3\mathbf{r} d^3\mathbf{r}' \rho^{(2)}(\mathbf{r}, \mathbf{r}') \frac{G}{|\mathbf{r} - \mathbf{r}'|}. \quad (18)$$

After making the variable changes  $\mathbf{r} \rightarrow R\mathbf{q}$ ,  $\mathbf{r}' \rightarrow R\mathbf{q}'$ , and using the scaling behaviour (17) for  $\rho^{(2)}(R\mathbf{q}, R\mathbf{q}')$ , we find that  $\langle V_N \rangle / N$  is indeed an intensive quantity in the SL given by formula (12) with

$$u(\varepsilon, \eta) = -\frac{9}{32\pi^2} \int_{q \leq 1, q' \leq 1} d^3\mathbf{q} d^3\mathbf{q}' g^{(2)}(\mathbf{q}, \mathbf{q}'; \varepsilon, \eta) \frac{1}{|\mathbf{q} - \mathbf{q}'|}. \quad (19)$$

### 3.3 Fluctuations of potential energy

Let us now consider the fluctuations of the potential energy around its average. Similarly to formula (18), such fluctuations can be rewritten as

$$\begin{aligned} \langle V_N^2 \rangle - [\langle V_N \rangle]^2 &= \frac{1}{2} \int_{\Lambda^2} d^3\mathbf{r} d^3\mathbf{r}' \rho^{(2)}(\mathbf{r}, \mathbf{r}') \frac{G^2 m^2}{|\mathbf{r} - \mathbf{r}'|^2} + \int_{\Lambda^3} d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' \rho^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'') \frac{G^2 m}{|\mathbf{r} - \mathbf{r}'| |\mathbf{r} - \mathbf{r}''|} \\ &+ \frac{1}{4} \int_{\Lambda^4} d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' d^3\mathbf{r}''' \left[ \rho^{(4)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''') - \rho^{(2)}(\mathbf{r}, \mathbf{r}') \rho^{(2)}(\mathbf{r}'', \mathbf{r}''') \right] \frac{G^2}{|\mathbf{r} - \mathbf{r}'| |\mathbf{r}'' - \mathbf{r}'''|} \end{aligned} \quad (20)$$

where  $\rho^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$  and  $\rho^{(4)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''')$  are the three- and four-body mass distributions. In the scaling limit, the first two terms in expression (20) can be estimated by replacing  $\rho^{(2)}(\mathbf{r}, \mathbf{r}')$  and  $\rho^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$  by their scaling behaviours in terms of  $g^{(2)}(\mathbf{q}, \mathbf{q}'; \varepsilon, \eta)$  and  $g^{(3)}(\mathbf{q}, \mathbf{q}', \mathbf{q}''; \varepsilon, \eta)$ . The first term is then found to be of order  $N^0$ , while the second one behaves as  $N$ .

If we estimate the third term in expression (20) by using only the scaling behaviours of the involved mass distributions, we obtain not surprisingly a  $N^2$ -behaviour. However such an estimation should over-estimate the exact behaviour, because the involved correlations  $[\rho^{(4)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''') - \rho^{(2)}(\mathbf{r}, \mathbf{r}') \rho^{(2)}(\mathbf{r}'', \mathbf{r}''')]$  can reasonably be expected to become rather small compared to  $\rho^4$ , for spatial configurations  $(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''')$  describing a large part of the integration domain  $\Lambda^4$ . A precise estimation cannot be performed at this level, so here we only assume that the correlation contribution (54) should grow slower than  $N^2$ .

The previous simple arguments suggest that the square of fluctuations (20) should become small compared to  $N^2$  in the SL, *i.e.*

$$\langle V_N^2 \rangle - [\langle V_N \rangle]^2 \sim o(N^2). \quad (21)$$

Notice that the corresponding fluctuations for an ordinary system with short-range interactions at thermodynamical equilibrium, do satisfy the behaviour (21) since they are proportional to  $N$ .

### 3.4 Scaling decomposition ansatz

In the following, we will have to perform averages of quantities involving the potential energy  $V_N$ . Taking into account the extensivity of its average, as well as the behaviour (21) of its fluctuations, we propose to decompose  $V_N$  as

$$V_N \rightarrow \langle V_N \rangle + W_N, \quad (22)$$

with  $\langle V_N \rangle = O(N)$  and  $W_N = o(N)$  in the SL. We assume that such decomposition ansatz holds for most probable configurations which mainly determine the averages of interest, so it should provide the exact behaviours in the cases studied further.

Notice that both the extensivity of  $\langle V_N \rangle$  and the fluctuations behaviour (21), although quite plausible, are not well established at this level. In fact, we will show *a posteriori* that the mass distributions and correlations inferred from the decomposition ansatz, are such that the *a priori* assumptions about  $\langle V_N \rangle$  and  $(\langle V_N^2 \rangle - [\langle V_N \rangle]^2)$  are indeed satisfied. This provides some kind of consistency check of the derivations, but of course it does not constitute a proof. Moreover, in the calculations of some averages, subtle correlations involving  $V_N$  and other observables might occur, so the above decomposition ansatz would no longer work, even if  $\langle V_N \rangle$  and  $(\langle V_N^2 \rangle - [\langle V_N \rangle]^2)$  do behave as expected.

## 4 Emergence of local thermalization

In this section, we study the inhomogeneous mass density  $\rho(\mathbf{r})$  given by average (13) for energies per particle  $\varepsilon > -1/2$ . First, we express  $\rho(\mathbf{r})$  in terms of the gravitational potential created by all the particles except one which is fixed at position  $\mathbf{r}$ . Then, in the SL, we rewrite exactly the corresponding spatial average by exploiting both the extensivity and  $\varepsilon > -1/2$ . Applying the decomposition ansatz (22) to that exact expression, we obtain a Boltzmann-like formula for  $\rho(\mathbf{r})$  in the SL, with some temperature which emerges naturally. We conclude by a few comments, in particular about the corresponding formula for the one-body distribution  $f^{(1)}(\mathbf{r}, \mathbf{p})$  in phase space.

### 4.1 Introduction of the gravitational potential

As a starting point, we compute the average (13) with the microcanonical distribution  $f_{\text{micro}}$  given by expression (3). Then, the standard integration over the momenta of the  $N$  particles leads to

$$\rho(\mathbf{r}) = B(E, N, \Lambda) \int_{\Lambda^{N-1}, |\mathbf{r}_i - \mathbf{r}_j| > \sigma} \prod_{i=2}^N d^3 \mathbf{r}_i [E - V_N(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N)]^{3N/2-1} \times \theta(E - V_N(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N)). \quad (23)$$

In that expression,  $\theta(\xi)$  is the usual Heaviside function such that  $\theta(\xi) = 1$  for  $\xi > 0$  and  $\theta(\xi) = 0$  for  $\xi < 0$ , while the normalization constant  $B(E, N, \Lambda)$  ensures that the spatial integral of  $\rho(\mathbf{r})$  over the box does provide the total mass  $Nm$  of the system. The conditions  $|\mathbf{r}_i - \mathbf{r}_j| > \sigma$ , arising from the hard sphere interaction, apply to any pair of particles, in particular those including particle 1 fixed at  $\mathbf{r}_1 = \mathbf{r}$ .

For  $\varepsilon > -1/2$ , it turns out that  $(E - V_N(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N))$  is positive for any spatial configuration thanks to the upper bound (11) for the potential energy. Then, the Heaviside function can be replaced by 1 in the expression (23). That simplification is crucial for further transformations. In particular, let us introduce the gravitational potential  $\Phi(\mathbf{r}|\mathbf{r}_2, \dots, \mathbf{r}_N)$  at  $\mathbf{r}$  created by the  $(N-1)$  particles located at  $\mathbf{r}_2, \dots, \mathbf{r}_N$ . According to the decomposition

$$V_N(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) = V_{N-1}(\mathbf{r}_2, \dots, \mathbf{r}_N) + m\Phi(\mathbf{r}|\mathbf{r}_2, \dots, \mathbf{r}_N), \quad (24)$$

we can exactly rewrite formula (23) as

$$\rho(\mathbf{r}) = B(E, N, \Lambda) \int_{\Lambda^{N-1}} d\mu_{N-1} \prod_{i=2}^N \theta(|\mathbf{r}_i - \mathbf{r}|/\sigma - 1) \times [E - V_{N-1}]^{3/2} \left[ 1 - \frac{m\Phi}{E - V_{N-1}} \right]^{3N/2-1}, \quad (25)$$

where  $d\mu_{N-1}$  denotes the unnormalized microcanonical measure for the spatial configurations  $(\mathbf{r}_2, \dots, \mathbf{r}_N)$  of a system made of  $(N-1)$  particles, enclosed in the same spherical box with volume  $\Lambda$  and the same energy  $E$  as the genuine system with  $N$  particles,

$$d\mu_{N-1} = \prod_{i=2}^N d^3\mathbf{r}_i \prod_{i<j} \theta(|\mathbf{r}_i - \mathbf{r}_j|/\sigma - 1) [E - V_{N-1}(\mathbf{r}_2, \dots, \mathbf{r}_N)]^{3(N-1)/2-1}. \quad (26)$$

Notice that, if  $\varepsilon < -1/2$ , then  $V_{N-1}$  and  $\Phi$  are coupled through the constraint  $E - V_{N-1} - m\Phi \geq 0$ , so the formula (25) *a priori* no longer holds.

## 4.2 Exploiting the extensivity of potential energy

Let us consider identity

$$\left[1 - \frac{m\Phi}{E - V_{N-1}}\right]^{3N/2-1} = \exp\left\{(3N/2-1) \ln\left[1 - \frac{m\Phi}{E - V_{N-1}}\right]\right\}. \quad (27)$$

Since  $m\Phi = O(1)$  and  $E - V_{N-1} = O(N)$ , we can expand the logarithm in powers of  $m\Phi/(E - V_{N-1})$ . After multiplication by factor  $(3N/2-1)$ , we see that the linear term provides a contribution of order  $O(1)$ , while higher powers provide vanishing contributions when  $N \rightarrow \infty$ . Accordingly, we obtain for any spatial configuration

$$\left[1 - \frac{m\Phi}{E - V_{N-1}}\right]^{3N/2-1} \sim \exp\left\{-\frac{3Nm\Phi}{2(E - V_{N-1})}\right\} \quad (28)$$

in the SL. Inserting that asymptotic behaviour inside the r.h.s. of formula (25), we infer in the SL

$$\rho(\mathbf{r}) \sim B(E, N, \Lambda) \int_{\Lambda^{N-1}} d\mu_{N-1} \prod_{i=2}^N \theta(|\mathbf{r}_i - \mathbf{r}|/\sigma - 1) \times [E - V_{N-1}]^{3/2} \exp\left\{-\frac{3Nm\Phi}{2(E - V_{N-1})}\right\}. \quad (29)$$

We stress that asymptotic expression (29) is exact, provided that  $\varepsilon > -1/2$ . The extensivity of the potential energy for any spatial configuration, which follows from the particular scaling defined here, plays a crucial role in the derivation of that behaviour. Within other scalings, which do not preserve that remarkable extensivity property, formula (29) would break down.

## 4.3 Boltzmann-like formula

If we introduce the kinetic energy  $K_{N-1}$  of the  $(N-1)$  particles for a given spatial configuration, the exponential factor on the r.h.s. of the asymptotic expression (29), can be seen as some kind of Boltzmann factor. However, at this level, the corresponding temperature fluctuates. Here, we show that such fluctuations can be neglected by applying the fluctuation ansatz described in Section 3.

The integral with the measure  $d\mu_{N-1}$  in the r.h.s. of formula (29) is proportional to the microcanonical average of the quantity

$$\prod_{i=2}^N \theta(|\mathbf{r}_i - \mathbf{r}|/\sigma - 1) [E - V_{N-1}]^{3/2} \exp\left\{-\frac{3Nm\Phi}{2(E - V_{N-1})}\right\} \quad (30)$$

for the system with  $(N-1)$  particles. If we introduce the corresponding microcanonical average

$$\langle V_{N-1} \rangle_{N-1} = \frac{\int_{\Lambda^{N-1}} d\mu_{N-1} V_{N-1}}{\int_{\Lambda^{N-1}} d\mu_{N-1}} \quad (31)$$

of  $V_{N-1}$ , we can rewrite

$$E - V_{N-1} = E - \langle V_{N-1} \rangle_{N-1} - W_{N-1}, \quad (32)$$

where  $W_{N-1}$  denotes the deviation of  $V_{N-1}$  with respect to its average for a given spatial configuration. Now, according to the fluctuation ansatz, we assume that for the most probable configurations which provide the main contributions to the average of quantity (30),  $W_{N-1}$  remains small compared to  $N$ , so

$$[E - V_{N-1}]^{3/2} = [E - \langle V_{N-1} \rangle_{N-1}]^{3/2} [1 + o(1)], \quad (33)$$

and

$$\frac{3N}{2[E - V_{N-1}]} = \frac{3N}{2[E - \langle V_{N-1} \rangle_{N-1}]} + o(1). \quad (34)$$

Moreover, the average potential energy  $\langle V_{N-1} \rangle_{N-1}$  for the system with  $(N-1)$  particles differs by a term of order  $O(1)$  from its counterpart  $\langle V_N \rangle$  for the genuine system with  $N$  particles. Thus, if we define the microcanonical temperature  $T(E, N, A)$  through the usual relation

$$E - \langle V_N \rangle = \langle K_N \rangle = \frac{3NT}{2}, \quad (35)$$

we eventually find in the SL

$$\rho(\mathbf{r}) \sim B(E, N, A) \left[ \frac{3NT(E, N, A)}{2} \right]^{3/2} \int_{A^{N-1}} d\mu_{N-1} \times \prod_{i=2}^N \theta(|\mathbf{r}_i - \mathbf{r}|/\sigma - 1) \exp \left\{ -\frac{m\Phi}{T(E, N, A)} \right\}. \quad (36)$$

The presence of the usual Boltzmann factor  $\exp(-m\Phi/T)$  in the asymptotic behaviour (36) does not mean that the whole system is in contact with a thermostat. We recall that the system is isolated with a well-defined energy  $E$ , as illustrated by the presence in behaviour (36) of the microcanonical measure  $d\mu_{N-1}$  which defines the integration over spatial configurations. However, the emergence of temperature  $T(E, N, A)$  can be traced back to some thermalization at a local level as argued and exploited in the next section. That temperature is indeed an intensive positive quantity in the SL, namely  $T(E, N, A)$  reduces to  $Gm_0^2/\ell_0$  times a dimensionless function  $T^*(\varepsilon, \eta)$  when  $N \rightarrow \infty$  while  $\varepsilon$  and  $\eta$  are the parameters which remain fixed. The interpretation in terms of a local thermodynamical equilibrium is confirmed by the analysis of the one-body distribution  $f^{(1)}(\mathbf{r}, \mathbf{p})$  in phase space. Starting from the corresponding microcanonical definition, and using again both the extensivity of the potential energy and the fluctuation ansatz, we find that  $f^{(1)}(\mathbf{r}, \mathbf{p})$  can be rewritten, in the SL, as  $\rho(\mathbf{r})$  times the Maxwell distribution proportional to  $\exp(-\mathbf{p}^2/(2mT))$ .

Eventually, we point out that the asymptotic formula (36) relies on the fluctuation ansatz, the validity of which might break down for some sets of parameters  $(\varepsilon, \eta)$  which determine the stationary state of the infinite system. Formula (36) might then fail for states with large collective fluctuations, as discussed in the next section.

## 5 Hydrostatic approach

### 5.1 Emergence of local equilibrium and separation of scales

The emergence of a local thermodynamical equilibrium at temperature  $T$ , can be interpreted as resulting from the collisions between the hard spheres<sup>2</sup>. The gravitational interactions between the particles inside a finite volume surrounding  $\mathbf{r}$ , can be safely neglected in the considered scaling limit since the mass of each particle then vanishes. In particular, notice that the gravitational interaction energy between two spheres at contact,  $-Gm^2/\sigma$ , vanishes as  $N^{-2/3}$  in the SL, so it becomes small compared to the thermal kinetic energy  $T$  which remains finite. Thus, the local equilibrium is entirely determined by the hard-core interactions, namely by the local packing fraction  $\eta(\mathbf{r}) = \eta\rho(\mathbf{r})/\rho$ .

---

<sup>2</sup> Before the SL is taken, namely for  $N$  finite, the gravitational interactions between close particles inside a small volume surrounding  $\mathbf{r}$  should also contribute to the thermalization process.



According to the above argument, at the local level around point  $\mathbf{r}$ , gravitation should intervene only through the gravitational potential

$$\phi(\mathbf{r}) = - \int_A d^3\mathbf{r}' \rho(\mathbf{r}') \frac{G}{|\mathbf{r}' - \mathbf{r}|} \quad (37)$$

created by the whole mass distribution  $\rho(\mathbf{r}')$ , which can be viewed as some external potential for the corresponding local system. In the expression (36), this amounts to replace the fluctuating potential  $\Phi$  by  $\phi(\mathbf{r})$ , namely to apply some kind of fluctuation ansatz to  $\Phi$  by assuming that its fluctuations vanish in the SL.

Within the present picture, the typical length scales become completely separated in the SL. On the one hand, the density  $\rho(\mathbf{r})$  is expected to vary on a length scale of order  $R$ , as made explicit through the scaling behaviour (15). On the other hand, the local correlation length  $\lambda(\mathbf{r})$  is controlled by hard sphere interactions, so it takes the form [21,22]

$$\lambda(\mathbf{r}) = \sigma \xi_{\text{HS}}(\eta(\mathbf{r})) \quad (38)$$

where  $\xi_{\text{HS}}(\eta)$  is some dimensionless function of  $\eta$ . In the SL with  $\mathbf{q} = \mathbf{r}/R$  fixed, since  $\eta(R\mathbf{q})$  goes to a finite value,  $\lambda(R\mathbf{q})$  behaves as  $\sigma$  and vanishes as  $N^{-2/15}$ . Thus, the density indeed varies on a scale infinitely larger than the local correlation length.

## 5.2 The coupled hydrostatic and gravitational equations

The emergence of a local thermodynamical equilibrium, combined to the separation of length scales, lead us to write down the hydrostatic equilibrium equation

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r}) \nabla \phi(\mathbf{r}) , \quad (39)$$

which expresses the balance between pressure and gravitational forces. Furthermore, in that equation, the local pressure  $P(\mathbf{r})$  reduces to that of an homogeneous gas of pure hard spheres without gravitational interactions, at temperature  $T$  and number density  $\rho(\mathbf{r})/m$ , i.e.

$$P(\mathbf{r}) = \frac{T\rho(\mathbf{r})}{m} p_{\text{HS}}(\eta\rho(\mathbf{r})/\rho) , \quad (40)$$

where  $p_{\text{HS}}$  is the dimensionless hard-sphere pressure which depends only on the local packing frac-

tion  $\eta(\mathbf{r}) = \eta\rho(\mathbf{r})/\rho$ .

The present analysis strongly suggests that the density profile in the SL is *a priori* exactly determined by the coupled equations (37) and (39), with pressure  $P$  replaced by its hard-sphere expression (40), together with the total mass constraint

$$\int_A d^3\mathbf{r} \rho(\mathbf{r}) = Nm , \quad (41)$$

and the total energy constraint

$$\frac{3NT}{2} + \frac{1}{2} \int_A d^3\mathbf{r} \rho(\mathbf{r}) \phi(\mathbf{r}) = E . \quad (42)$$

Notice that all those equations and constraints can be recast in terms of the dimensionless variable  $\mathbf{q} = \mathbf{r}/R$  and of the dimensionless density profile  $g^{(1)}(q; \varepsilon, \eta) = \lim_{\text{SL}} \rho(R\mathbf{q})/\rho$ . We recall that temperature  $T$  is not a given parameter, so the corresponding dimensionless temperature  $T^*(\varepsilon, \eta) = T/(Gm_0^2/\ell_0)$  has to be determined through the resolution of the whole system of equations where  $\varepsilon$  and  $\eta$  are the fixed control parameters.

If the gravitational long-ranged interactions can be treated as a mean-field level through the introduction of the average potential  $\phi(\mathbf{r})$ , the hard-core short-ranged interactions play a crucial role in the determination of the density profile. Thus, the celebrated mean-field theory for point particles [17, 18, 11] , does not provide the exact density profile in the present SL. The corresponding equations can

be retrieved from the present analysis by replacing in the hydrostatic equation (39) the hard sphere pressure by its ideal counterpart. After an obvious integration, this provides the Boltzmann expression

$$\rho(\mathbf{r}) = C \rho \exp \left\{ -\frac{m\phi(\mathbf{r})}{T} \right\} , \quad (43)$$

where  $C$  is some dimensionless normalization constant which remains to be determined. The resulting mean-field coupled equations for  $\rho(\mathbf{r})$  and  $\phi(\mathbf{r})$  can be also recast in terms of the dimensionless variable  $\mathbf{q} = \mathbf{r}/R$  and of the dimensionless density profile  $\rho(R\mathbf{q})/\rho$ . Now  $\varepsilon$  is the sole control parameter, and it can be rewritten as  $E/(GM^2/R)$  where the total mass is  $M = Nm$ . The properties inferred from that so-called model of the isothermal sphere have been widely studied in the literature [23]. We stress that the presence of the hard-core interactions should invalidate, even at a qualitative level, some mean-field predictions. Indeed, since the local packing fraction  $\eta(\mathbf{r}) = \eta\rho(\mathbf{r})/\rho$  cannot exceed [24,25] the maximal packing fraction  $\eta_{\max} \simeq 0.7405\dots$ , the exact density  $\rho(\mathbf{r})$  is bounded everywhere by  $\rho\eta_{\max}/\eta$ , while some mean-field features are directly related to a possible unbounded increase of the core density  $\rho(0)$ .

### 5.3 Consistency checks

The derivations which ultimately provide the coupled equations (37) and (39), involve various *a priori* assumptions. For consistency purposes, it is of course necessary to check such assumptions within the global picture which sustains the hydrostatic approach.

A first assumption concerns the extensivity property (12) of  $\langle V_N \rangle$ , namely the existence of a well-defined potential energy per particle  $u(\varepsilon, \eta)$  in the SL. Within the hydrostatic approach,  $\langle V_N \rangle$  reduces to the self-gravitational energy of a sphere with mass density  $\rho(\mathbf{r})$ ,

$$V_{\text{self}} = \frac{1}{2} \int_A d^3\mathbf{r} \rho(\mathbf{r}) \phi(\mathbf{r}) = -\frac{1}{2} \int_{A^2} d^3\mathbf{r} d^3\mathbf{r}' \rho(\mathbf{r}) \rho(\mathbf{r}') \frac{G}{|\mathbf{r} - \mathbf{r}'|} . \quad (44)$$

Since  $\rho(\mathbf{r})$  does satisfy the scaling property (15),  $V_{\text{self}}$  is indeed extensive in the SL. However, notice that the exact expression (18) of  $\langle V_N \rangle$  can be recast as

$$\langle V_N \rangle = V_{\text{self}} + V_{\text{corr}} , \quad (45)$$

with

$$V_{\text{corr}} = -\frac{1}{2} \int_{A^2} d^3\mathbf{r} d^3\mathbf{r}' \rho^{(2,T)}(\mathbf{r}, \mathbf{r}') \frac{G}{|\mathbf{r} - \mathbf{r}'|} \quad (46)$$

and the truncated mass distribution or mass correlations,

$$\rho^{(2,T)}(\mathbf{r}, \mathbf{r}') = \rho^{(2)}(\mathbf{r}, \mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}') . \quad (47)$$

If the coupled equations (37) and (39) do not give access to the two-body mass correlations, the existence of a local thermodynamical equilibrium entirely determined by hard sphere interactions allow us to introduce a simple description of such correlations. Because of the hard core,  $\rho^{(2,T)}(\mathbf{r}, \mathbf{r}')$  reduces to  $-\rho(\mathbf{r})\rho(\mathbf{r}')$  for  $|\mathbf{r} - \mathbf{r}'| < \sigma$ . Moreover, we expect a decay of  $\rho^{(2,T)}(\mathbf{r}, \mathbf{r}')$  over the hard-sphere local correlation length  $\lambda(\mathbf{r})$ . As argued above,  $\lambda(R\mathbf{q})$  becomes proportional to  $\sigma$  in the SL. Thus, the contribution of the vicinity of point  $\mathbf{r} = R\mathbf{q}$  to

$$\int_A d^3\mathbf{r}' \rho^{(2,T)}(\mathbf{r}, \mathbf{r}') \frac{G}{|\mathbf{r} - \mathbf{r}'|} \quad (48)$$

is of order  $G\rho^2\sigma^2 = O(N^{-4/15})$  in the SL. The remaining contribution to the integral (48) of points  $\mathbf{r}'$  such that  $|\mathbf{r} - \mathbf{r}'| \gg \sigma$  can be roughly estimated by replacing  $\rho^{(2,T)}(\mathbf{r}, \mathbf{r}')$  by a constant times  $m\rho(\mathbf{r})/\Lambda$  which does not depend on  $\mathbf{r}'$ . That spread homogeneous approximation is inspired by the sum rule

$$\int_A d^3\mathbf{r}' \rho^{(2,T)}(\mathbf{r}, \mathbf{r}') = -m\rho(\mathbf{r}) , \quad (49)$$

which follows from particle conservation. The corresponding contribution to integral (48) is then of order  $Gm\rho R^2/\Lambda = O(N^{-3/5})$  which becomes small compared to that of the region  $|\mathbf{r} - \mathbf{r}'| \sim \sigma$ . Accordingly, we find that the correlation energy (46) is of order  $G\rho^2 R^3 \sigma^2 = O(N^{1/3})$ . Thus,  $\langle V_N \rangle$  is indeed extensive, and the potential energy per particle is entirely given by the self part  $V_{\text{self}}$ , namely

$$u(\varepsilon, \eta) = -\frac{9}{32\pi^2} \int_{q \leq 1, q' \leq 1} d^3\mathbf{q} d^3\mathbf{q}' g^{(1)}(\mathbf{q}; \varepsilon, \eta) g^{(1)}(\mathbf{q}'; \varepsilon, \eta) \frac{1}{|\mathbf{q} - \mathbf{q}'|}. \quad (50)$$

Let us consider now the fluctuations  $(\langle V_N^2 \rangle - [\langle V_N \rangle]^2)$  of the potential energy, which are given by formula (20). We can rewrite the first two terms as mean-field like terms

$$\frac{1}{2} \int_{\Lambda^2} d^3\mathbf{r} d^3\mathbf{r}' \rho(\mathbf{r}) \rho(\mathbf{r}') \frac{G^2 m^2}{|\mathbf{r} - \mathbf{r}'|^2} \quad (51)$$

and

$$\int_{\Lambda^3} d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' \rho(\mathbf{r}) \rho(\mathbf{r}') \rho(\mathbf{r}'') \frac{G^2 m}{|\mathbf{r} - \mathbf{r}'| |\mathbf{r} - \mathbf{r}''|}, \quad (52)$$

plus the corresponding correlation terms associated with  $[\rho^{(2)}(\mathbf{r}, \mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}')] \rho(\mathbf{r}'')$  and  $[\rho^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'') - \rho(\mathbf{r})\rho(\mathbf{r}')\rho(\mathbf{r}'')]$ . Since  $\rho(\mathbf{r})$  does satisfy the scaling property (15), the mean-field contributions (51) and (52) are of order  $N^0$  and  $N$  respectively. The corresponding two- and three-body correlation contributions are readily estimated within a simple modelization of the truncated distributions analogous to the one used above for analyzing contribution (46) to the average  $\langle V_N \rangle$  itself. They are found to become small compared to their mean-field counterparts in the SL.

It remains to estimate the contribution of the third term in expression (20) of the fluctuations  $(\langle V_N^2 \rangle - [\langle V_N \rangle]^2)$ . If we define

$$n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''') = \rho^{(4)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'', \mathbf{r}''') / \rho^{(2)}(\mathbf{r}, \mathbf{r}') - \rho^{(2)}(\mathbf{r}'', \mathbf{r}'''), \quad (53)$$

we can rewrite that four-body correlation term as

$$\frac{1}{4} \int_{\Lambda^2} d^3\mathbf{r} d^3\mathbf{r}' \rho^{(2)}(\mathbf{r}, \mathbf{r}') \frac{G}{|\mathbf{r} - \mathbf{r}'|} \int_{\Lambda^2} d^3\mathbf{r}'' d^3\mathbf{r}''' n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''') \frac{G}{|\mathbf{r}'' - \mathbf{r}'''|}. \quad (54)$$

Similarly to the case of  $\rho^{(2, \text{T})}(\mathbf{r}, \mathbf{r}')$ , we expect that, for two given points  $\mathbf{r}$  and  $\mathbf{r}'$ ,  $n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''')$  takes non-vanishing values for spatial configurations such that one or more relative distances  $|\mathbf{r}'' - \mathbf{r}|$ ,  $|\mathbf{r}'' - \mathbf{r}'|$ ,  $|\mathbf{r}''' - \mathbf{r}|$ ,  $|\mathbf{r}''' - \mathbf{r}'|$  is of order  $\sigma$ . The corresponding contributions to the integral

$$\int_{\Lambda^2} d^3\mathbf{r}'' d^3\mathbf{r}''' n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''') \frac{G}{|\mathbf{r}'' - \mathbf{r}'''|} \quad (55)$$

are then readily estimated along similar lines as above when analyzing the two-body correlation term (48). For each of the four regions when one of the points  $\mathbf{r}''$  or  $\mathbf{r}'''$  is close to either  $\mathbf{r}$  or  $\mathbf{r}'$ , we find a contribution to integral (55) of order  $G\rho^2 \sigma^3 R^2 = O(1)$ . After integration over  $\mathbf{r}$  and  $\mathbf{r}'$ , the corresponding contribution to the term (54) is  $O(N)$ . All the other spatial configurations  $(\mathbf{r}'', \mathbf{r}''')$  for which hard-sphere correlations determine  $n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''')$ , ultimately provide contributions  $o(N)$  to (54). Eventually, it remains to determine the contributions of regions such that all distances  $|\mathbf{r}'' - \mathbf{r}|$ ,  $|\mathbf{r}'' - \mathbf{r}'|$ ,  $|\mathbf{r}''' - \mathbf{r}|$ ,  $|\mathbf{r}''' - \mathbf{r}'|$ , are large compared to  $\sigma$ . As for the case of  $\rho^{(2, \text{T})}(\mathbf{r}, \mathbf{r}')$ , we again use a spread homogeneous approximation inspired by the sum rule

$$\int_{\Lambda} d^3\mathbf{r}'' d^3\mathbf{r}''' n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''') = 2(3 - 2N)m^2, \quad (56)$$

namely we replace  $n_{\mathbf{r}, \mathbf{r}'}^{(2)}(\mathbf{r}'', \mathbf{r}''')$  by a constant proportional to  $Nm^2/\Lambda^2$ . This provides a contribution of order 1 to integral (55), and ultimately a contribution  $O(N)$  to (54). Accordingly, the four-body correlation term (54) is of order  $N$ , so fluctuations  $(\langle V_N^2 \rangle - [\langle V_N \rangle]^2)$  indeed grow slower than  $N^2$ , in agreement with the assumption (21).

## 5.4 Further limitations

We stress that there are other implicit assumptions in the hydrostatic approach, which might break down, at least for some sets of values for the control parameters  $(\varepsilon, \eta)$ . In particular, if the local packing fraction exceeds some critical value  $\eta_{LC}$ , the system is expected to undergo a phase transition from a liquid to a crystalline phase [21, 22]. Then, the density  $\rho(\mathbf{r})$  would oscillate with a spatial period of order  $\sigma$  in the bulk, for  $\mathbf{r} = R\mathbf{q}$  with  $|\mathbf{q}| < 1$ . Therefore, the hydrostatic equation which assumes local homogeneity on a large scale compared to  $\sigma$  should be no longer valid<sup>3</sup>. This might occur for sufficiently low values of  $\varepsilon$  and not too small values of  $\eta$ . On the contrary, for not too low values of  $\varepsilon$ , and  $\eta$  sufficiently small, we can expect that the local packing fraction  $\eta(\mathbf{r})$  never exceeds the critical value  $\eta_{LC}$ , so the system remains in a fluid phase everywhere as implicitly assumed in the hydrostatic approach.

Another important assumption in the derivations relies on the existence of a single most probable state which provides the leading contributions to the averages of the quantities of interest, while the corresponding corrections are associated with fluctuations which remain relatively small. Within the inferred hydrostatic approach, this means that the solution of the coupled equations (37) and (39) has to exist and to be unique. Such existence and uniqueness are in fact not guaranteed. This is well illustrated by the analysis of the point-like version of those equations with  $\eta = 0$ : there are no solutions for  $\varepsilon < \varepsilon_{\min} \simeq -0.335$  [18], while two and more solutions can coexist in some range  $\varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$  [23]. For  $\eta \neq 0$ , numerical studies [26, 27] using approximate simple forms of the hard-sphere equation of state  $p_{HS}(\eta)$  indicate that some of the previous features are still observed, so they are not specific to the point-like case  $\eta = 0$ . In fact, they are related to the attractive nature of gravitational interactions which favors the collapse of the system. The absence of solutions to the coupled equations (37) and (39) might be related to the flatness of the microcanonical distribution. When two or more solutions coexist, one would expect some kind of multimodal shape for that distribution. In any of such cases, there appears macroscopic fluctuations which cannot be neglected with respect to the averages themselves. Thus, the whole scheme leading to the hydrostatic approach should then break down. A proper many-body microcanonical analysis of the system, which might then undergo some phase transitions [23], remains a challenging and rather difficult problem.

The tendency to collapse induces either the violation of the condition  $\eta(\mathbf{r}) < \eta_{LC}$  near the core  $r = 0$ , or the lack of the uniqueness property of the hydrostatic solution. That mechanism is of course related to Jeans instability [28], which is controlled by the Jeans length  $L_J \propto (T/(G\rho m))^{1/2}$ . Within the SL,  $L_J$  reduces to the size  $R$  of the system times  $[T^*(\varepsilon, \eta)/3]^{1/2}$ . When  $\varepsilon \rightarrow \infty$  at a fixed  $\eta < \eta_{LC}$ , since the potential energy is always negative, the dimensionless temperature  $T^*(\varepsilon, \eta)$  also diverges. More precisely, the lower and upper bounds (10) and (11) on that potential energy, imply the behaviour  $T^*(\varepsilon, \eta) \sim 2\varepsilon/3$  when  $\varepsilon \rightarrow \infty$  at a fixed  $\eta < \eta_{LC}$ . Accordingly, the ratio  $L_J/R$  also diverges, so previous limitations arising from the tendency to collapse should not occur. Thus, for  $\varepsilon$  sufficiently large and  $\eta$  sufficiently small, the hydrostatic approach can be reasonably expected to provide the exact density profile. In fact, the gravitational interactions can then be treated as a small perturbation, so the density  $\rho(\mathbf{r})$  should be almost uniform and close to  $\rho$ , in relation with the quite plausible limit behaviour  $g^{(1)}(\mathbf{q}; \varepsilon, \eta) \rightarrow 3/(4\pi)$  when  $\varepsilon \rightarrow \infty$  with  $q$  and  $\eta < \eta_{LC}$  fixed. When  $\varepsilon$  is decreased, temperature  $T^*(\varepsilon, \eta)$  should decrease so the ratio  $L_J/R$  should also decrease. When  $L_J$  becomes of the order of the size  $R$ , or even smaller than  $R$ , Jeans-like instabilities should play an important role and invalidate the assumptions underlying the hydrostatic approach. Therefore, there should exist some threshold value  $\varepsilon_c(\eta)$  depending on  $\eta$ , below which the hydrostatic approach fails. Notice that, if the derivation of the hydrostatic approach has been carried out with the condition  $\varepsilon > -1/2$ , this does not mean that  $\varepsilon_c(\eta)$  is necessarily smaller than  $-1/2$ , as suggested by the point-like case  $\eta = 0$  for which no mean-field solutions exist for  $\varepsilon < \varepsilon_{\min} \simeq -0.335$ .

---

<sup>3</sup> Notice that close to the wall of the spherical box, i.e. for  $r$  almost equal to  $R$ , the mass density  $\rho(r)$  varies on a scale  $\sigma$ . The corresponding shape does not show in the SL of  $\rho(qR)$  for fixed  $q < 1$ .

## 6 Conclusion

We have presented in this paper a novel approach to the statistical mechanics of self-gravitating particles in the microcanonical ensemble which emphasizes that gravitational interactions can be treated at the mean-field level. The main idea is that, while the gravitational constant is not rescaled, an appropriate scaling of the parameters of the particle allows us to define a non-trivial equilibrium state with the usual H-stability and extensivity properties. If this scaling is introduced through simple physical requirements, it turns out that, rather unexpectedly, it also ensures the spontaneous emergence of the local thermodynamical equilibrium (43).

Several comments and open questions are however important to draw at this stage.

i) We show here that, locally, the gas of self-gravitating particles is thermalized through collisions, so the present scaling defines a temperature. Moreover this finding justifies a posteriori some previous attempts to study self-gravitating particles within a canonical ensemble [3,5]. However, that thermalization is expected to occur only at sufficiently large enough energy per particle. Furthermore, although the thermalization is recovered at the local level, there is still no clear definition of an external thermostat at the macroscopic level, with which the system is exchanging heat or kinetic energy. Indeed, as it is well known, the energy being conserved, a gain (or loss) of kinetic energy can be obtained by a decrease (or increase) of potential energy. It is still debated nowadays [29] how one can use a canonical framework with a clear physical meaning.

ii) The hard-core regularization is essential, on the one hand for avoiding the collapse of the finite system [6], and on the other hand for ensuring through collisions the emergence of a local equilibrium in the scaling limit. Once that limit has been taken, hard-core effects intervene in the local pressure which ultimately determines the density profile, at least for a large enough energy per particle. At this level, the hard core can be removed and we recover the density profile for point particles determined within the familiar mean-field description of the isothermal sphere.

iii) We have considered throughout the paper a large enough energy per particle  $\varepsilon$ , but of course states with negative  $\varepsilon$  are of strong interest. Indeed, we have discovered that positive values prevent the Jeans length to be small enough and thus hinder any equilibrium collapsed state. A non vanishing fraction of matter will stay on the boundary of the system, even if the number of particles  $N$  diverges to infinity. Statistically, a large part of these particles will have momenta pointing outwards, leading to the partial evaporation of the system. This effect is not described within the present model in which all particles are maintained artificially into the spherical domain. To avoid this problem, one might consider negative energies, which would lead to sufficiently small Jeans length to constrain particles to stay far from the boundaries: the fraction of evaporating particles might therefore be vanishingly small. Interestingly, this effect is reminiscent of two particles in a Keplerian system, which are restricted to confined elliptic trajectories only for negative energies.

Both long-range and short-range effects, as well as confinement, can therefore be traced back in the results derived within this new scaling approach to statistical mechanics in the microcanonical ensemble. They are at the origin of the difficulties, but also of the interests of this fascinating problem. We are however convinced that this new scaling approach is an important step forward for understanding the complete statistical mechanics treatment encompassing the possibility to tackle phase transitions and fragmentations of a system of self-gravitating particles.

**Acknowledgements** This work has been supported by the contract LORIS (ANR-10-CEXC-010-01).

## References

1. Joyce, M.: Infinite self-gravitating systems and cosmological structure formation. In: Campa, A., Giansanti, A., Morigi, G., Sylos Labini, F. (eds.) *Dynamics and Thermodynamics of Systems with Long-range Interaction: Theory and Experiments*. American Institute of Physics, 237-268 (2008).
2. Thirring, W.: Systems with negative specific heat, *Zeitschrift für Physik* 235, 339 (1970).
3. Kiessling, M. K. H.: On the equilibrium statistical mechanics of isothermal classical self-gravitating matter, *Journal of Statistical Physics* 55, 203 (1989).
4. Padmanabhan, T.: Statistical mechanics of gravitating systems, *Physics Reports* 188, 285 (1990).
5. Chavanis, P. H.: Phase transitions in self-gravitating systems, *International Journal of Modern Physics B* 20, 3113 (2006).

6. Pomeau, Y.: Statistical mechanics of gravitational plasmas. In: Cichocki, B., Napiórkowski, M., Piasecki J. (eds.). 2nd Warsaw School of Statistical Physics, Warsaw University Press (2008).
7. Ruelle, D.: Statistical Mechanics: Rigorous Results, Benjamin, New York (1969).
8. Messer, J., Spohn, H.: Statistical mechanics of the isothermal lane-Emden equation, *Journal of Statistical Physics* 29, 561 (1982).
9. Kiessling, M. K. H., Lebowitz, J. L.: The micro-canonical point vortex ensemble: Beyond equivalence, *Letters in Mathematical Physics* 42, 43 (1997).
10. Barré, J., Mukamel, D., Ruffo, S.: Inequivalence of ensembles in a system with long-range interactions, *Physical Review Letters* 87, 030601 (2001).
11. Lynden-Bell, D., Wood, R.: Gravo-thermal catastrophe in isothermal spheres and onset of red-giant structure for stellar systems, *Monthly Notices of the Royal Astronomical Society* 138, 495 (1968).
12. Henon, M.: L'évolution initiale d'un amas sphérique, *Annales d'Astrophysique* 27, 83 (1964).
13. Antoni, M., Ruffo, S., Torcini, A.: First-order microcanonical transitions in finite mean-field models, *Europhysics Letters* 66, 645-651 (2004).
14. Chavanis P.H.: On the lifetime of metastable states in self-gravitating systems, *Astronomy and Astrophysics* 432, 117-138 (2005).
15. Posch, H. A., Thirring, W.: Some aspects of the classical three-body problem that are close or foreign to physical intuition, *Journal of Mathematical Physics* 41, 3430 (2000).
16. Chabanol, M.L., Corson, F., Pomeau, Y.: Statistical mechanics of point particles with a gravitational interaction, *Europhysics Letters* 50, 148 (2000).
17. Emden, R.: *Gaskugeln*, Teubner, Leipzig (1907).
18. Antonov, V. A. *Vest. Leningrad Gos. Univ.* 7, 135 (1962). English translation in *IAU Symposium* 113, *Dynamics of Globular Clusters*, (ed.) Goodman, J. Hut, P.: Dordrecht: Reidel, pp. 525-540 (1985).
19. de Vega, H.J., Sánchez, N.: Statistical mechanics of the self-gravitating gas: I. Thermodynamic limit and phase diagrams, *Nuclear Physics* 625, 409-459 (2002).
20. Votyakov, E.V., Hidmi, H. I., De Martino, A., Gross, D. H.: Microcanonical mean-field thermodynamics of self-gravitating and rotating systems, *Physical Review Letters*, 89, 031101 (2002).
21. Hansen, J.P., McDonald, I.R., *Theory of Simple Liquids*, Academic Press (2006).
22. Mulero, A. (ed.), *Lecture notes in Physics* 753, Springer (2008).
23. Chavanis, P. H.: Statistical mechanics of two-dimensional vortices and stellar systems. In: Dauxois, T., Ruffo, S., Arimondo, E., Wilkens, M. (eds.) *Dynamics and Thermodynamics of Systems with Long-Range Interactions*, *Lecture Notes in Physics* 602, Springer (2002).
24. Hales, T. C.: An overview of the Kepler conjecture, *Annals of Mathematics* 162, 1065-1185 (2005).
25. Hales, T.C., Harrison, J., McLaughlin, S., Nipkow, T., Obua, S., Zumkeller, R., A Revision of the Proof of the Kepler Conjecture, *Discrete & Computation Geometry* 44, 1-34 (2010).
26. Aronson, E. B., Hansen, C.J.: Thermal equilibrium states of a classical system with gravitation, *Astrophysical Journal* 177, 145 (1972).
27. Stahl, B., Kiessling, M. K. H., Schindler, K.: Phase transition in gravitation systems and the formation of condensed objects, *Planetary and Space Sciences* 43, 271 (1995).
28. Jeans, J. H.: The Stability of a Spherical Nebula, *Philosophical Transactions of the Royal Society of London. Series A* 199, 1(1902).
29. Dauxois, T., Ruffo, S., Arimondo, E., Wilkens, M.: Dynamics and Thermodynamics of Systems with Long Range Interactions: an Introduction. In: Dauxois, T., Ruffo, S., Arimondo, E., Wilkens, M. (eds.) *Dynamics and Thermodynamics of Systems with Long-Range Interactions*, *Lecture Notes in Physics* 602, Springer (2002).